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On semi cover-avoiding subgroups of finite groups

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Abstract

A subgroup H of a finite group G is said to have the semi cover-avoiding property in G if there is a normal series of G such that H covers or avoids every normal factor of the series. In this paper, some new results are obtained based on the assumption that some subgroups have the semi cover-avoiding property in the group.

MSC: 20D20

1. Introduction

All groups considered in this paper are finite.

In 1962, Gaschütz introduced a certain conjugacy class of subgroups of a solvable group [9]. These subgroups have the cover-avoiding property, that is, they do not only avoid the complemented chief factors of the solvable group but also cover the rest of its chief factors. Thereafter, many authors continued to study this property (see, for example, [10, 17]). In these papers, the main aim was to find some kind of subgroups of a solvable group having the cover-avoiding property. However, the natural question arises whether we can obtain structural insight into a group when some of its subgroups have the cover-avoiding property.

In 1993, Ezquerro gave some characterizations for a group G to be p-supersolvable and supersolvable under the assumption that all maximal subgroups of some Sylow subgroups of G have the cover-avoiding property [7]. Recently, Guo and Shum pushed further this approach and obtained some characterizations for a solvable group and a p-solvable group based on the assumption that some of its subgroups have the cover-avoiding property [11]. More recently, in [8], Fan, Guo and Shum introduced the semi cover-avoiding property, which covers not only the cover-avoidance property but also c-normality, and obtained some new results. It has been proved that the semi cover-avoiding property is suitable for describing the structure of groups. In this paper, we will continue to study the influence of some subgroups having the semi cover-avoiding property on the structure of groups.

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2. Basic definitions and preliminary results

Let G be a group. The *Frattini subgroup* of G is defined to be the intersection of all the maximal subgroups of G, with the convention that the Frattini subgroup equals G when G does not have a maximal subgroup. The Frattini subgroup, which is obviously characteristic, is denoted by $\Phi(G)$. When M and N are normal subgroups of the group G with $N \leq M$, then the quotient group M/N is called a normal factor of G. It is clear that every chief factor of a group is a normal factor of the group. A subgroup H of G is said to *cover* M/N if $HM = HN$. On the other hand, if $H \cap M = H \cap N$, then H is said to *avoid* M/N. It is easy to check the following fact.

Lemma 2.1 (*[8, Lemma 1]*). *Let H be a subgroup of a group G and let $1 < \cdots < N < \cdots < M < \cdots < G$ be a normal series of G. If H covers M/N, then H covers any quotient factor between N and M of any refinement of the normal series. Likewise, if H avoids M/N, then H avoids any quotient factor between N and M of any refinement of the normal series.*

Definition 2.2. Let H be a subgroup of a group G.

(1) The group H is said to have the cover-avoiding property in G if for every chief factor M/N of G, H either covers M/N or avoids M/N.
(2) The group H is said to have the semi cover-avoiding property in G if there is a normal series $1 = G_0 < G_1 < \cdots < G_t = G$ of G such that, for every $j = 1, 2, \ldots, t$, H either covers G_j/G_{j-1} or avoids G_j/G_{j-1}.

Remark 2.3. Let H be a subgroup of a group G.

(1) If H has the semi cover-avoiding property in G then, by Lemma 2.1, there is a chief series of G such that H covers or avoids every chief factor of the series.
(2) It is clear by definition that, if H has the semi cover-avoiding property, the order of H is the product of the orders of the covered normal factors in the series.

It is clear that H must have the semi cover-avoiding property in G if H has the cover-avoiding property in G. However, the converse is not true.

Example 2.4. Let A_4 be the alternating group of degree 4 and $C_2 = \langle c\rangle$ a cyclic group of order 2, generated by an element written as c; let $G = C_2 \times A_4$. Then $A_4 = K_4 \cdot \langle t\rangle$, where $K_4 = \langle a, b\rangle$ is the Klein four group with generators a and b of order 2 and $\langle t\rangle$ is a cyclic group of order 3. Take $H = \langle ac\rangle$ to be the subgroup of G generated by ac. It is easy to see that the following series

$$1 < K_4 < A_4 < C_2 \times A_4 = G$$

is a chief series of G and that H covers G/A_4 and avoids the rest. That is, H has the semi cover-avoiding property in G. However,

$$HC_2 = \langle a, c\rangle \neq H \cdot (K_4 \times C_2)$$

and

$$H \cap (K_4 \times C_2) = H \neq 1 = H \cap C_2.$$

Thus H does not have the cover-avoiding property in G since $(K_4 \times C_2)/C_2$ is a chief factor of G.

The following lemma is useful for us in proving our results.

Lemma 2.5. *Let H be a subgroup of the group G. If H has the semi cover-avoiding property in G, then H has the semi cover-avoiding property in K for every subgroup K of G with $H \leq K$.*

Proof. Assume that

$$1 = G_0 < G_1 < \cdots < G_t = G$$

is a chief series of G such that, for every $j = 1, 2, \ldots, t$, H either covers G_j/G_{j-1} or avoids G_j/G_{j-1}. Then

$$1 = G_0 \cap K \leq G_1 \cap K \leq \cdots \leq G_t \cap K = K$$

is a normal series of K. For every $j = 1, 2, \dots, t$, if H covers G_j/G_{j-1}, then it follows from $G_jH = G_{j-1}H$ that $H(G_j \cap K) = H(G_{j-1} \cap K)$; if H avoids G_j/G_{j-1}, then it follows from $G_j \cap H = G_{j-1} \cap H$ that $H \cap (G_j \cap K) = H \cap (G_{j-1} \cap K)$. Therefore H has the semi cover-avoiding property in K. This completes the proof. □

By [8, Lemma 2], we have the following fact, which is also useful for us.

Lemma 2.6. *Let N be a normal subgroup of the group G and let H be a subgroup of G having the semi cover-avoiding property in G. Then HN/N has the semi cover-avoiding property in G/N if one of the following holds:*

(1) $N \le H$
(2) $gcd(|H|, |N|) = 1$, *where $gcd(-,-)$ denotes the greatest common divisor.*

Lemma 2.7 (*[13, Lemma 2.6]*). *Let N be a solvable normal subgroup of a group G with $N \neq 1$. If every minimal normal subgroup of G which is contained in N is not contained in the Frattini subgroup $\Phi(G)$ of G, then the fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G which are contained in N.*

Recall that a formation of groups is a class of groups $\mathcal{F}$ which is closed under homomorphic images and is such that $G/(M \cap N)$ belongs to $\mathcal{F}$ whenever M, N are normal subgroups of a group G such that G/M and G/N belong to $\mathcal{F}$. We call a formation $\mathcal{F}$ *saturated* if G belongs to $\mathcal{F}$ whenever $G/\Phi(G)$ belongs to $\mathcal{F}$.

Let Π be the set of all prime numbers. A function f defined on Π is called a *formation function* if $f(p)$, possibly empty, is a formation for all $p \in \Pi$. A chief factor H/K of a group G is called f-*central* in G if $G/C_G(H/K)$ belongs to $f(p)$ for every prime number p dividing the order of H/K. A formation $\mathcal{F}$ is then called a *local* formation if there exists a formation function f such that $\mathcal{F}$ is the class of all groups G for which every chief factor of G is f-central in G. When $\mathcal{F}$ is a local formation defined by a formation function f, we write $\mathcal{F} = LF(f)$ and we call f a *local definition* of $\mathcal{F}$.

It is known that, among all possible local definitions for a local formation $\mathcal{F}$, there exists exactly one of them, which we denote by F, such that the formation function F is both integrated (that is, $F(p) \subseteq \mathcal{F}$ for all $p \in \Pi$) and full (that is, $\mathcal{N}_pF(p) = F(p)$ for all $p \in \Pi$), where $\mathcal{N}_p$ is the class of p-groups.

It is also well-known that a formation $\mathcal{F}$ is saturated if and only if $\mathcal{F}$ is a local formation (see [6]).

Lemma 2.8 (*[6, Proposition IV. 3.11]*). *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where F_i is both an integrated and full formation function of $\mathcal{F}_i$ $(i = 1, 2)$. Then the following statements are equivalent.*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,
(2) $F_1(p) \subseteq F_2(p)$ *for all* $p \in P$.

Lemma 2.9 (*[1, Lemma 2]*). *Let $\mathcal{F}$ be a saturated formation. Let G be a group which does not belong to $\mathcal{F}$, and suppose that there exists a maximal subgroup M of G such that M belongs to $\mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the fitting subgroup. Then $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G, $G^{\mathcal{F}}$ is a p-group for some prime p, and $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, $G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

3. Main results

In 1970, Buckley proved that a group of odd order is supersolvable if all its minimal subgroups are normal [4]. Later, Srinivasan showed that a group G is supersolvable if all maximal subgroups of all Sylow subgroups of G are normal [15]. These two important results on supersolvable groups have been generalized by many authors. Now we will investigate the structure of the groups in which every maximal subgroup of its Sylow p-subgroups has the semi cover-avoiding property and also the groups in which every minimal subgroup of its p-Sylow subgroups has the semi cover-avoiding property for some prime p. We shall generalize a number of known results.

We begin with the following.

Lemma 3.1. *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let P be a p-Sylow subgroup of G. If there is a maximal subgroup P_1 of P such that P_1 has the semi cover-avoiding property in G, then G is p-solvable.*

Proof. By Definition 2.2 and Remark 2.3(1), there exists a chief series of G such that P_1 covers or avoids every chief factor of the series. By Remark 2.3(2), there is only one chief factor of the series, whose order is divisible by p and not covered by P_1. Then the p-Sylow subgroup of this chief factor has order p and is therefore cyclic. Now by our hypothesis and Burnside's theorem [12, Theorem IV.2.6 and 2.7], this chief factor is p-nilpotent and therefore of order p. Hence G is p-solvable. This completes the proof. □

Theorem 3.2. *Let p be the smallest prime dividing the order of the group G and let P be a p-Sylow subgroup of G. If P is cyclic or every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Proof. If P is a cyclic group then, again by the theorem of Burnside [12, Theorem IV.2.6 and 2.7], G is p-nilpotent. So we may assume that P is not a cyclic group. Thus G is p-solvable by Lemma 3.1.

Now we claim that for any maximal subgroup P_1 of P, G has a normal subgroup T_{P_1} such that $[G : T_{P_1}] = p$ and $P_1 \leq T_{P_1}$. In fact, since P_1 has the semi cover-avoiding property in G, there exists a chief series of G such that P_1 covers or avoids every chief factor of the series. By Remark 2.3(2), there is only one chief factor of the series whose order is divisible by p and not covered by P_1. By an inductive argument one can assume that this factor is at the bottom of the chief series. Thus this factor is a minimal normal subgroup N of G of order p. By Gaschütz's Theorem [12, Theorem I.17.4], there exists a complement R of N in G containing P_1. On the other hand, the minimality of p implies that N is contained in the center of G. Hence R is a normal subgroup of G with index p and contains P_1.

Let M be the intersection of all these normal subgroups T_{P_1} of G. Then M has p-power index in G and the intersection M and P is the Frattini subgroup of P. By a famous theorem of Tate (see [3]), G is p-nilpotent. This completes the proof of the theorem. □

Corollary 3.3. *Let p be the smallest prime dividing the order of the group G, N a normal subgroup of G such that G/N is p-nilpotent, and let P be a p-Sylow subgroup of N. If P is cyclic or has the property that every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Proof. By Lemma 2.5, P is cyclic or every maximal subgroup of P has the semi cover-avoiding property in N. By Theorem 3.2, N is p-nilpotent. Let H be a normal Hall p'-subgroup of N. Then H is normal in G. Consider the quotient group G/H. It is clear that N/H is normal in G/H and $(G/H)/(N/H) \simeq G/N$ is p-nilpotent. In view of Lemma 2.6, we conclude that G/H satisfies the hypotheses of our corollary for its normal subgroup N/H. If $H \neq 1$ then, by induction, we have that G/H is p-nilpotent and so G is p-nilpotent. Hence we assume $H = 1$ and therefore $N = P$ is a p-group. For any prime q dividing the order of G with $q \neq p$ and $Q \in \mathrm{Syl}_q(G)$, it is clear that PQ is a subgroup of G and hence PQ is p-nilpotent by Theorem 3.2; we conclude that $PQ = P \times Q$. Let K/N be the normal p-complement of G/N. The above arguments imply that $K = N \times K_1$ where K_1 is a Hall p'-subgroup of K. Hence K_1 is a normal p-complement of G. This completes the proof. □

Remark 3.4. It is clear that the assumption that p is the smallest prime dividing the order of G in Theorem 3.2 and Corollary 3.3 is essential. In fact, the symmetric group S_3 of degree 3 is a counterexample for $p = 3$. However, we have the following results.

Theorem 3.5. *Let p be an odd prime dividing the order of the group G and P a p-Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is not true and let G be a counterexample of smallest order. Then $O_{p'}(G) = 1$. In fact, if $O_{p'}(G) \neq 1$, then we can consider the quotient group $G/O_{p'}(G)$. By Lemma 2.6, $G/O_{p'}(G)$ satisfies the hypotheses of our theorem. Thus, by the minimality of G, we see that $G/O_{p'}(G)$ is p-nilpotent and therefore G is p-nilpotent, which is a contradiction. Furthermore, we claim the following facts.

(1) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is obvious that $N_M(P) \leq N_G(P)$, and hence $N_M(P)$, is p-nilpotent. Applying Lemma 2.5 we see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(2) $O_p(G) \neq 1$ and G is p-solvable.

Since G is not p-nilpotent, by a result of Thompson [16, Corollary], there exists a characteristic subgroup $H(\neq 1)$ of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $H < K \leq P$. Since H is characteristic in P and P is normal in $N_G(P)$, we have that H is normal in $N_G(P)$ and so $N_G(H) \geq N_G(P)$. Assertion (1) implies that $N_G(H) = G$. It is easy to see that $O_p(G) = H$. By the choice of H, we know that $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P satisfying $O_p(G) < K \leq P$. Now, exploiting the result of Thompson [16, Corollary] again, we see that $G/O_p(G)$ is p-nilpotent and that, therefore, G is p-solvable.

We now make use of the above claims to prove our theorem. Let N be a minimal normal subgroup of G. Then $N \leq O_p(G)$ and, by Lemma 2.6, the quotient group G/N satisfies the hypotheses of our theorem. Now, by the minimality of G, we see that G/N is p-nilpotent. Furthermore, since the class of all p-nilpotent groups is a saturated formation, we may assume that N is the unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Thus, by Lemma 2.7, we know that $O_p(G) = N$. Let P_1 be a maximal subgroup of P. Then by the uniqueness of N, P_1 covers or avoids $N/1$. If P_1 avoids $N/1$, then N is a cyclic group of order p. It follows that $N \leq Z(P)$ and therefore $P \leq C_G(N)$. However, by [14, Theorem 9.3.1], $C_G(N) \leq N$. Hence $N = P$ is a p-Sylow subgroup of G. By the hypotheses, $G = N_G(N)$ is p-nilpotent, a contradiction. So we may assume that every maximal subgroup of P covers $N/1$. Hence $N \leq \Phi(P)$. It follows from [14, Theorem 5.2.13] that $N \leq \Phi(G)$, the final contradiction. This completes the proof of the theorem. □

Corollary 3.6. *Let p be an odd prime number dividing the order of the group G and N a normal subgroup of G such that G/N is p-nilpotent. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent, where P is a p-Sylow subgroup of N.*

Proof. It is clear that $N_N(P)$ is p-nilpotent and that every maximal subgroup of P has the semi cover-avoiding property in N. By Theorem 3.5, N is p-nilpotent. Now, let H be the normal Hall p'-subgroup of N. Then H is normal in G. By the arguments used in the proof of Corollary 3.3, we may assume that $H = 1$ and $N = P$ is a p-group. In this case, by our hypotheses, $N_G(P) = G$ is p-nilpotent. □

If we replace the hypothesis that $N_G(P)$ is p-nilpotent in Theorem 3.5 by the hypothesis that G is p-solvable, we obtain the following result.

Theorem 3.7. *Let p be a prime dividing the order of the p-solvable group G and let P be a p-Sylow subgroup of G. If P is a cyclic group or every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample of smallest order. If P is cyclic then G is already p-supersolvable, a contradiction. Now suppose that P is not a cyclic group. Let N be a minimal normal subgroup of G. Then N is a p-group or p'-group. By Lemma 2.6, the quotient group G/N satisfies the hypotheses of our theorem. Now, by the minimality of G, we see that G/N is p-supersolvable. Furthermore, since the class of p-supersolvable groups is a saturated formation, we may assume that N is the unique minimal normal subgroup of G. If N is a p'-group, then G is already p-supersolvable. So we may assume that N is a p-group. If N is contained in every maximal subgroup of P, then $N \leq \Phi(G)$. It follows from the p-supersolvability of G/N that G is p-supersolvable. Hence there is a maximal subgroup P_1 of P such that P_1 does not contain N. By the hypotheses, P_1 avoids $N/1$ and therefore N is a cyclic group of order p. It follows that G is p-supersolvable. This completes the proof. □

Given a group G, the existence of a minimal subgroup of G having the semi cover-avoiding property has structural implications for G. We will now explore some of these.

We first establish the following theorem.

Theorem 3.8. *Let p be the smallest prime number dividing the order of the group G and let P be a p-Sylow subgroup of G such that every minimal subgroup of P has the semi cover-avoiding property in G. When $p = 2$ suppose, in addition, that either every cyclic subgroup of P of order 4 also has the semi cover-avoiding property in G or that P is quaternion-free. Then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of smallest order. Then G is not p-nilpotent. As all p-Sylow subgroups of G are conjugate in G, it is easy to see that, in view of Lemma 2.5, every subgroup H of G satisfies the hypotheses of the theorem whenever p is a prime factor of $|H|$. Therefore G is a non-p-nilpotent group but every proper subgroup of G is p-nilpotent. By the result of Itô [14, Theorem 10.3.3], we see that G cannot be nilpotent but every proper subgroup of G is nilpotent. By a result of Schmidt [14, Theorem 9.1.9 and Exercises 9.1.11], there exists a p-Sylow subgroup P of G and a q-Sylow subgroup Q of G such that $G = PQ$ where P is normal in G and Q is cyclic. Moreover, $P = G'$ and P is of exponent p when p is odd and of exponent at most 4 when $p = 2$. Furthermore, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Let $x \in P$ but $x \notin \Phi(P)$. Then x is of order p or 4.

Case 1. p is odd or $p = 2$ and every cyclic subgroup of P of order 4 also has the semi cover-avoiding property in G. In this case, there exists a chief series of G

$$1 = G_0 < G_1 < \cdots < G_t = G$$

such that $\langle x\rangle$ covers or avoids every G_j/G_{j-1}. Since $x \in G$, for some k, $x \notin G_k$ but $x \in G_{k+1}$. It follows from $G_k \cap \langle x\rangle \neq G_{k+1} \cap \langle x\rangle$ that $G_k\langle x\rangle = G_{k+1}\langle x\rangle = G_{k+1}$. Hence G_{k+1}/G_k is a cyclic group of order p or 4. The normality of $P \cap G_k$ implies that $(P \cap G_k)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we see that $(P \cap G_k)\Phi(P) = \Phi(P)$ or P. If $(P \cap G_k)\Phi(P) = P$, then $P \cap G_k = P$, contradicting $x \notin P \cap G_k$. Thus $P \cap G_k \le \Phi(P)$. Consider the normal subgroup $(P \cap G_{k+1})\Phi(P)$. By using the similar arguments as the above, we have that $(P \cap G_{k+1})\Phi(P) = \Phi(P)$ or P. It follows that $(P \cap G_{k+1})\Phi(P) = P$ since $x \notin \Phi(P)$ but $x \in P \cap G_{k+1}$. Thus $P/\Phi(P)$ is a cyclic group of order p. Noting that p is the smallest prime dividing the order of G, we see that $G/\Phi(P)$ is p-nilpotent and therefore G is p-nilpotent, a contradiction.

Case 2. $p = 2$ and P is quaternion-free. In this case, if x is of order 2 then, since x has the semi cover-avoiding property in G, we may still use the arguments in Case 1 to get a contradiction. So we may assume that every element of P of order 2 is contained in $\Phi(P)$ and therefore $\Phi(P) \neq 1$. Noticing that $Z(P) \cap (\Omega_1(\Phi(P))) \le Z(G)$, we see that $Z(G) \cap P \cap G^{\mathcal{N}} = Z(G) \cap P \cap G' = Z(G) \cap P \neq 1$. However, by applying [5, Theorem 2.8], we have $P \cap G^{\mathcal{N}} \cap Z(G) = 1$, a contradiction (where $G^{\mathcal{N}}$ is the nilpotent residual of G). Hence G is p-nilpotent and the proof of the theorem is complete. □

Corollary 3.9. *Let p be the smallest prime dividing the order of the group G and P a p-Sylow subgroup of G. Suppose that every minimal subgroup of $P \cap O^p(G)$ has the semi cover-avoiding property in G and, when $p = 2$, suppose in addition that either every cyclic subgroup of $P \cap O^p(G)$ of order 4 also has the semi cover-avoiding property in G or that $P \cap O^p(G)$ is quaternion-free. Then G is p-nilpotent, where $O^p(G)$ is the smallest normal subgroup of G such that $G/O^p(G)$ is a p-group.*

Proof. Since $P \cap O^p(G)$ is a p-Sylow subgroup of $O^p(G)$, by Lemma 2.5 and Theorem 3.8, $O^p(G)$ is p-nilpotent and therefore G is p-nilpotent since $G/O^p(G)$ is a p-group. □

Remark 3.10. The hypothesis that p is the smallest prime dividing the order of the group G in Theorem 3.8 and Corollary 3.9 cannot be removed. For example, if we let G be the symmetric group S_3 on three letters and $p = 3$, then we see that the statements of Theorem 3.8 and Corollary 3.9 do not hold.

Given a group G, observing that $O^p(H) \le O^p(G)$ for every subgroup H of G and using Lemma 2.5 and Corollary 3.9, we obtain at once the following result.

Corollary 3.11. *Suppose that, for every prime p dividing the order of the group G, there is a p-Sylow subgroup P of G such that every minimal subgroup of $P \cap O^p(G)$ has the semi cover-avoiding property in G and that, when $p = 2$, either every cyclic subgroup of $P \cap O^p(G)$ of order 4 also has the semi cover-avoiding property in G or that $P \cap O^p(G)$ is quaternion-free. Then G is a Sylow tower group of supersolvable type.*

Theorem 3.12. *Let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$ and let N be a normal subgroup of the group G such that G/N belongs to $\mathcal{F}$. Suppose that, for every prime p dividing the order of N, there is a p-Sylow subgroup P of N such that every minimal subgroup of $P \cap O^p(G)$ has the semi cover-avoiding property* *in G and that, when $p = 2$, either every cyclic group of order 4 of $P \cap O^p(G)$ also has the semi cover-avoiding property in G or that $P \cap O^p(G)$ is quaternion-free. Then G belongs to $\mathcal{F}$.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.5 and Corollary 3.11, we see that N is a Sylow tower group of supersolvable type. Let q be the largest prime dividing the order of N and let Q be a q-Sylow subgroup of N. Then Q is normal in N and therefore is normal in G; furthermore, every minimal subgroup of $Q \cap O^q(G)$ has the semi cover-avoiding property in G and, when $q = 2$, every cyclic group of order 4 of $Q \cap O^2(G)$ also has the semi cover-avoiding property in G or $Q \cap O^2(G)$ is quaternion-free. Now we consider the quotient group G/Q. Since $(G/Q)/(N/Q) \simeq G/N$ and $O^p(G/Q) = O^p(G)/Q$ for every prime $p \neq q$, we know that G/Q satisfies the hypotheses of the theorem by Lemma 2.6. Thus, the minimality of G implies that G/Q is in $\mathcal{F}$.

Now, when G is not in $\mathcal{F}$, the $\mathcal{F}$-residual $G^{\mathcal{F}}$ of G is nontrivial. Since $G/O^q(G)$ is a q-group and since therefore $G/O^q(G)$ belongs to $\mathcal{F}$, necessarily $G/Q \cap O^q(G)$ belongs to $\mathcal{F}$ as well. It follows that $G^{\mathcal{F}}$ is contained in $Q \cap O^q(G)$. By [2, Theorem 3.5], there exists a maximal subgroup M of G such that $G = MF'(G)$ where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$ and $G/\mathrm{core}_G(M)$ is not in $\mathcal{F}$. Then $G = MG^{\mathcal{F}}$ and therefore $G = MF(G)$ since $G^{\mathcal{F}}$ is a q-group where $F(G)$ is the fitting subgroup of G. It is now clear that, in view of Lemma 2.5, M satisfies the hypotheses of the theorem for its normal subgroup $M \cap Q$. Hence the minimality of G implies that M must be in $\mathcal{F}$.

By Lemma 2.9, $G^{\mathcal{F}}$ has exponent q when $q \neq 2$ and exponent at most 4 when $q = 2$. If $G^{\mathcal{F}}$ is an elementary abelian group, then $G^{\mathcal{F}}$ is a minimal normal subgroup of G. For any minimal subgroup A of $G^{\mathcal{F}}$, we know that A has the semi cover-avoiding property in G since $G^{\mathcal{F}}$ is contained in $Q \cap O^q(G)$, and therefore there exists a chief series of G

$$1 = G_0 < G_1 < \cdots < G_t = G$$

such that A covers or avoids every G_j/G_{j-1}. Since $A \leq G$, for some k, $A \nleq G_k$ but $A \leq G_{k+1}$. It follows from $G_k \cap A \neq G_{k+1} \cap A$ that $G_k A = G_{k+1} A = G_{k+1}$. Hence G_{k+1}/G_k is a cyclic group of order q. Since $G^{\mathcal{F}} \cap G_{k+1}$ is a normal subgroup in G and since $G^{\mathcal{F}}$ is a minimal normal subgroup of G, we have $G^{\mathcal{F}} \leq G_{k+1}$. Similarly, we have $G^{\mathcal{F}} \cap G_k = 1$. It follows that $G_{k+1} = G_k G^{\mathcal{F}}$ and therefore $G^{\mathcal{F}} \simeq G_{k+1}/G_k$ is a cyclic group of order q. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is G-isomorphic to $\mathrm{Soc}(G/\mathrm{core}_G(M))$, it follows that $G/\mathrm{core}_G(M)$ is supersolvable, a contradiction.

If $G^{\mathcal{F}}$ is not an elementary abelian group, then $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group by Lemma 2.9. Let $x \in G^{\mathcal{F}}$ but $x \notin \Phi(G^{\mathcal{F}})$. Then x is of order q or 4. If q is odd or $q = 2$ and every cyclic subgroup of Q of order 4 also has the semi cover-avoiding property in G, then there exists a chief series of G

$$1 = G_0 < G_1 < \cdots < G_t = G$$

such that $\langle x \rangle$ covers or avoids every G_j/G_{j-1}. Since $x \in G$, for some k, $x \notin G_k$ but $x \in G_{k+1}$. It follows from $G_k \cap \langle x \rangle \neq G_{k+1} \cap \langle x \rangle$ that $G_k\langle x \rangle = G_{k+1}\langle x \rangle = G_{k+1}$. Hence G_{k+1}/G_k is a cyclic group of order q or 4. Since $G^{\mathcal{F}} \cap G_k$ is a normal subgroup in G and $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$, we have $(G^{\mathcal{F}} \cap G_k)\Phi(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}}$. If $(G^{\mathcal{F}} \cap G_k)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}}$, then $G^{\mathcal{F}} \cap G_k = G^{\mathcal{F}}$, contradicting $x \notin G^{\mathcal{F}} \cap G_k$. Thus $G^{\mathcal{F}} \cap G_k \leq \Phi(G^{\mathcal{F}})$. Consider the normal subgroup $(G^{\mathcal{F}} \cap G_{k+1})\Phi(G^{\mathcal{F}})$. Also since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$, we have $(G^{\mathcal{F}} \cap G_{k+1})\Phi(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}}$. It follows that $G^{\mathcal{F}} \cap G_{k+1}\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}}$ since $x \in G^{\mathcal{F}} \cap G_{k+1}$ but $x \notin \Phi(G^{\mathcal{F}})$. Hence $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a cyclic group of order q. As before, we run into a contradiction.

There remains the case where $q = 2$ and where $Q \cap O^q(G)$ is quaternion-free. Of course, every minimal subgroup of $Q \cap O^q(G)$ has the semi cover-avoiding property in G. Let R be an r-Sylow subgroup of G with $r \neq 2$ and $G_1 = RG^{\mathcal{F}}$. Then $G^{\mathcal{F}}$ is a 2-Sylow subgroup of G_1. Observing that $G^{\mathcal{F}} \leq Q \cap O^2(G)$, we have that G_1 is 2-nilpotent by Theorem 3.8. It follows that $G^{\mathcal{F}} \leq C_G(R)$ and therefore $Z(G) \cap G^{\mathcal{F}} \neq 1$. Since $G^{\mathcal{F}} \leq G^{\mathcal{N}}$, we have $Z(G) \cap G^{\mathcal{N}} \cap Q \neq 1$, in contradiction to [5, Theorem 2.8], where $G^{\mathcal{N}}$ is the nilpotent residual of G. This completes the proof of the theorem. □

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